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FREE GROUP ALGEBRAS IN MALCEV-NEUMANN SKEW FIELDS OF FRACTIONS

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Abstract. Let K be a skew field and $(G,<)$ an ordered group. We show that the skew field generated by the group ring $K[G]$ inside the Malcev-Neumann series ring $K((G;<))$ contains noncommutative free group algebras.

Introduction

There are two conjectures on the existence of free objects in skew fields that have attracted much attention. The first one was made by A. Lichtman [20]:

(G) If a skew field D is not commutative (i.e. not a field), then $D\setminus\{0\}$ contains a noncyclic free group.

An affirmative answer has been given when the center of D is uncountable [5] and when D is finite-dimensional over its center [13]. The second conjecture was formulated in [27]:

(A) Let D be a skew field with center Z. If D is finitely generated as skew field over Z, then D contains a noncommutative free Z-algebra.

Some important results have been obtained in [24], [25], [21], [2] and other papers. In many examples in which conjecture (A) holds, D in fact contains a noncommutative free group Z-algebra [10]. For example, this always happens if the center of D is uncountable [16]. Therefore it makes sense to consider the following unifying conjecture:

(GA) Let D be a skew field with center Z. If D is finitely generated as a skew field over Z and D is infinite dimensional over Z, then D contains a noncommutative free group Z-algebra.

For more details on these and related conjectures the reader is referred to [15].

We deal with conjectures (A) and (GA) for an important class of skew fields obtained from a skew field K and an ordered group $(G,<)$, i.e. G is a group and $<$ is a total order in G such that the relation $x<y$ implies that $zx<zy$ and $xz<yz$ for all $x,y,z\in G$. The Malcev-Neumann series ring $K((G;<))$ is defined to be the set of all formal series $f=\sum_{x\in G} xa_x$, with $a_x\in K$, whose *support* is a well-ordered subset of $(G,<)$ where $\operatorname{supp}(f)=\{x\in G\mid a_x\neq 0\}$. The operations of K and G, together with the relations $ax=xa$, for all $a\in K$ and $x\in G$, induce a multiplication in $K((G;<))$. With this product and componentwise addition, $K((G;<))$ is a skew field [28], [29]. Clearly $K((G;<))$ contains the group ring $K[G]$. We denote by $K(G)$ the skew subfield of $K((G;<))$ generated by $K[G]$. It was proved in [6] that $K(G)$ is infinite dimensional over its center when G is not commutative.

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Our main result is Theorem 6. It states that if $(G,<)$ is not commutative, then $K(G)$ satisfies conjecture (GA). As a consequence one obtains that any skew field generated by a group ring $K[G]$ with G a noncommutative orderable group and K a skew field satisfies conjecture (A), see Corollary 14. We remark that our methods exhibit the generators of the free group algebra when the order of the group is well known.

Although $K(G)$ does not depend on the structure of the ordered group $(G,<)$ [17], our proof of the main result strongly depends on it. Our proof also needs of richer structures than group rings and the Malcev-Neumann series rings already defined, namely crossed products and the Malcev-Neumann series they determine. These and related concepts are reviewed in the first section.

Section 2 deals with results that will be used to reduce the problem of finding free group algebras inside skew fields. The first one is a generalization of the fact that finding free (group) algebras over the prime subfield equivalent to finding free (group) algebras over any central subfield [26, Lemma 1]. The second one states that if our orderable group G contains a free monoid, then $K(G)$ contains a free group algebra. We also give some results describing the behavior of Malcev-Neumann series rings that will be used later. They may be well known, but we provide a proof because we find them of independent interest and we have not found them in the literature.

The core of the paper is Section 3. Here we give a proof of our main result. It is divided in three subsections according to what type of ordered group $(G,<)$ is. The proof for ordered groups of type 1 consists in showing that the problem can be reduced to torsion-free nilpotent groups. For these groups the result was proved in [10]. For ordered groups of type 2 and 3, we are able to find free monoids inside G.

In Section 4 we give some consequences from the main result and the methods we have developed for its proof.

1. Preliminaries

In this section we present the concepts and notation that will be used throughout the paper.

A *skew field* is a nonzero associative ring with identity element and such that every nonzero element has an inverse.

If R is a ring, a *skew field of fractions* of R is a skew field D together with a ring embedding $R \hookrightarrow D$ such that D is generated as a skew field by (the image of) R.

1.1. **Ordered groups.**

We recall standard material on ordered groups. For further details the reader is referred to any monograph on the subject such as [12]. We will sometimes use the results reviewed here without any further reference.

A group G is *orderable* if there exists a total order $<$ of G such that, for all $x,y,z \in G$,

$$x<y \quad \text{implies that} \quad zx<zy \text{ and } xz<yz. \tag{1.1}$$

In this event we say that $(G,<)$ is an *ordered group*.

Let $(G,<)$ be an ordered group. A subgroup H of G is *convex* if for any $x,z \in H$ and $y \in G$, the inequalities $x<y<z$ imply that $y \in H$. If a convex subgroup N is normal in G, then G/N can be ordered in a natural way: given $x,y \in G$ such that $xN \neq yN$, define $xN \prec yN$ if $x<y$. Moreover there is a natural bijection between the convex subgroups of $(G,<)$ that contain N and the convex subgroups of $(G/N,\prec)$.

An element $x \in G$ is *positive* if $1<x$.

The ordered group $(G,<)$ is *archimedean* if for any positive elements $x,y \in G$ there is a natural number n such that $x<y^n$, equivalently, $(G,<)$ has no other

convex subgroups other than $\{1\}$ and G. An archimedean group is order isomorphic to a subgroup of the additive group of the real numbers with the natural order.

If A, B are subgroups of the additive group of the real numbers with the natural order and $\varphi\colon A \to B$ is a *morphism of ordered groups* (i.e. a morphism of groups such that $x < y$ implies $\varphi(x) \leq \varphi(y)$), then there exists a real number $r \geq 0$ such that $\varphi(x) = rx$ for all $x \in A$.

The set Γ of convex subgroups of $(G, <)$ satisfies the following properties:

- Γ is a chain with respect to inclusion, that is, if $H_1, H_2 \in \Gamma$, then either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.
- The arbitrary union and intersection of elements in Γ belongs to Γ
- Γ is *infrainvariant*, that is, $xHx^{-1} \in \Gamma$ for each $H \in \Gamma$ and $x \in G$.

We say that a pair (N, H) is a *convex jump* if $N, H \in \Gamma$, $N \subsetneqq H$ and no subgroup in Γ lies properly between N and H. It is known that $N \lhd H$ for any convex jump (N, H). Hence H/N is order isomorphic to a subgroup of the additive group of the reals with the natural order.

Let $x \in G$. The *convex jump determined by* x is the convex jump (N_x, H_x), where N_x is the union of all convex subgroups of $(G, <)$ that do not contain x, and H_x is the intersection of all convex subgroups of $(G, <)$ that contain x.

1.2. **Crossed product group rings.** The following definitions and comments are from [30, Chapter 1].

Let R be a ring and let G be a group. A *crossed product* of G over R is an associative ring $R[\bar{G}; \sigma, \tau]$ which contains R and has as an R-basis the set $\bar{G}$, a copy of G. Thus each element of $R[\bar{G}; \sigma, \tau]$ is uniquely expressed as a finite sum $\Sigma_{x \in G} \bar{x} r_x$, with $r_x \in R$. Addition is componentwise and multiplication is determined by the two rules below. For $x, y \in G$ we have a *twisting*

$$\bar{x}\bar{y} = \overline{xy}\tau(x, y)$$

where $\tau\colon G \times G \to U(R)$, the group of units of R. Furthermore for $x \in G$ and $r \in R$ we have an *action*

$$r\bar{x} = \bar{x} r^{\sigma(x)}$$

where $\sigma\colon G \to \mathrm{Aut}(R)$ and $r^{\sigma(x)}$ denotes the image of r by $\sigma(x)$.

If $d\colon G \to U(R)$ assigns to each element $x \in G$ a unit d_x of R, then $\tilde{G} = \{\tilde{x} = \bar{x} d_x \mid x \in G\}$ is another R-basis for $R[\bar{G}; \sigma, \tau]$. Moreover there exist maps $\sigma'\colon G \to \mathrm{Aut}(R)$ and $\tau'\colon G \times G \to U(R)$ such that $R[\bar{G}; \sigma, \tau] = R[\tilde{G}; \sigma', \tau']$. We call this a *diagonal change of basis*. Via a diagonal change of basis if necessary, we will assume that $1_{R[\bar{G}; \sigma, \tau]} = \bar{1}$. The embedding of R into $R[\bar{G}; \sigma, \tau]$ is then given by $r \mapsto r\bar{1}$. Note that $\bar{G}$ acts on both $R[\bar{G}; \sigma, \tau]$ and R by conjugation and that for $x \in G$, $r \in R$ we have $\bar{x}^{-1} r \bar{x} = r^{\sigma(x)}$.

If there is no action or twisting, that is, if $\sigma(x)$ is the identity and $\tau(x, y) = 1$ for all $x, y \in G$, then $R[\bar{G}; \sigma, \tau] = R[G]$ is the usual *group ring*.

If N is a subgroup of G, the crossed product $R[\bar{N}; \sigma_{|N}, \tau_{|N \times N}]$ embeds in $R[\bar{G}; \sigma, \tau]$, and we will denote it as $R[\bar{N}; \sigma, \tau]$.

Suppose that N is a normal subgroup of G and let $S = R[\bar{N}; \sigma, \tau]$. Then $R[\bar{G}; \sigma, \tau]$ can be seen as a crossed product of G/N over S. More precisely $R[\bar{G}; \sigma, \tau] = S[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}]$, where $\overline{G/N}$, $\tilde{\sigma}$ and $\tilde{\tau}$ are defined as follows.

For each $1 \neq \alpha \in G/N$, let $x_\alpha \in G$ be a fixed representative of the class α. For the class $1 \in G/N$ choose $x_1 = 1 \in G$. Set $\bar{\alpha} = x_\alpha$ for all $\alpha \in G/N$. Then $G = \dot{\bigcup} \bar{\alpha} N$. Define $\overline{G/N} = \{\bar{\alpha} \mid \alpha \in G/N\} \subseteq \bar{G}$. This shows that $R[\bar{G}; \sigma, \tau] = \bigoplus_\alpha \bar{\alpha} S$. Therefore

$\overline{G/N}$ is an S-basis for $R[\bar{G};\alpha,\tau]$. Moreover, we have the maps:

$$\begin{array}{rl} \tilde{\sigma}\colon G/N \longrightarrow & \mathrm{Aut}(S) \\ \alpha \longmapsto & \tilde{\sigma}(\alpha)\colon S \longrightarrow S \\ & \qquad\quad y \longmapsto \bar{x}_\alpha^{-1} y \bar{x}_\alpha \end{array}$$

$$\begin{array}{rl} \tilde{\tau}\colon G/N \times G/N \longrightarrow & U(S) \\ (\alpha,\beta) \longmapsto & \tilde{\tau}(\alpha,\beta) = \bar{n}_{\alpha\beta}\tau(x_{\alpha\beta},n_{\alpha\beta})^{-1}\tau(x_\alpha,x_\beta) \end{array}$$

where $n_{\alpha\beta}$ is the unique element in N such that $x_{\alpha\beta}n_{\alpha\beta} = x_\alpha x_\beta$. Then $(\sum\limits_{n\in N}\bar{n}r_n)\bar{\alpha} = \bar{\alpha}(\sum\limits_{n\in N}\bar{n}r_n)^{\tilde{\sigma}(\alpha)}$ and $\bar{\alpha}\bar{\beta} = \overline{\alpha\beta}\tilde{\tau}(\alpha,\beta)$, as desired.

Notice that the assumption $(x_1 = 1)$ implies that S embeds in $R[\bar{G};\sigma,\tau]$ via $\bar{n}r \mapsto 1\cdot\bar{n}r$.

1.3. **Malcev-Neumann series rings.** Let R be a ring, and let $(G,<)$ be an ordered group. Form a crossed product $R[\bar{G};\sigma,\tau]$. We define a new ring, denoted $R((\bar{G};\sigma,\tau,<))$ and called *Malcev-Neumann series ring*, in which $R[\bar{G};\sigma,\tau]$ embeds. As a set

$$R((\bar{G};\sigma,\tau,<)) = \Big\{f = \sum_{x\in G}\bar{x}a_x \mid a_x\in R,\ \mathrm{supp}(f) \text{ is well ordered}\Big\},$$

where $\mathrm{supp}(f) = \{x\in G \mid a_x \neq 0\}$.

The addition and product are defined extending the ones in $R[\bar{G};\sigma,\tau]$. Thus, given $f = \sum\limits_{x\in G}\bar{x}a_x$ and $g = \sum\limits_{x\in G}\bar{x}b_x$ in $R((\bar{G};\sigma,\tau,<))$, the sum is defined by

$$f+g = \sum_{x\in G}\bar{x}(a_x+b_x),$$

and multiplication by

$$fg = \sum_{x\in G}\bar{x}\Big(\sum_{yz=x}\tau(y,z)a_y^{\sigma(z)}b_z\Big).$$

When the crossed product is a group ring $R[G]$, its Malcev-Neumann series ring will be denoted by $R((G;<))$.

Let $f\in R((\bar{G};\sigma,\tau,<))$ and $x_0 = \min\mathrm{supp}\, f$. If the coefficient of x_0 is an invertible element of R, then f is invertible. Hence, if R is a skew field, then $R((\bar{G};\sigma,\tau,<))$ is a skew field [28], [29].

If K is a skew field, the skew field of fractions of $K[\bar{G};\sigma,\tau]$ inside $K((\bar{G};\sigma,\tau,<))$ will be called the *Malcev-Neumann skew field of fractions* of $K[\bar{G};\sigma,\tau]$ and denoted by $K(\bar{G};\sigma,\tau)$. It is important to observe the following. For a subgroup H of G, $K((\bar{H};\sigma,\tau))$ and $K(\bar{H};\sigma,\tau)$ can be seen as skew subfields of $K((\bar{G};\sigma,\tau))$ and $K(\bar{G};\sigma,\tau)$, respectively, in the natural way. In the case of the group ring $K[G]$, the Malcev-Neumann skew field of fractions is denoted by $K(G)$. We remark that $K(\bar{G};\sigma,\tau)$ does not depend on the order $<$ of G, see [17].

2. Technical results

This section is devoted to prove some results that will be used to prove our main result, but they are interesting in themselves too.

The next lemma is a generalization of known results that we will obtain as a corollary. It reduces the problem of finding free (group) algebras over the center of a skew field. The proof is very similar to the one of the original result [26, Lemma 1].

We recall that a monoid is orderable if there exists a total order that satisfies condition (1.1). Hence ordered groups are ordered monoids.

Lemma 1. *Let R be a ring with prime subring P (i.e. generated by 1). Let M be a submonoid of $U(R)$ and let C be a subring of R. Suppose that the following conditions hold true*

1. *The algebra generated by P and M is the monoid algebra $P[M]$.*
2. *For each $c \in C$ and $f \in P[M]$, the equality $cf = 0$ implies that $c = 0$ or $f = 0$.*
3. *The elements of M commute with the elements of C.*
4. *M is an orderable monoid.*
5. *M embeds in some group H such that $H(M)$, the subgroup of H generated by M, has trivial center.*

Then the subring of R generated by C and M is the monoid ring $C[M]$.

Proof. Consider the multiplication map $\mu\colon C \otimes_P P[M] \to R$. By (3), to prove that $C[M]$ is contained in R, it is enough to show that μ is an injective map.

Suppose that $\mu(\sum_{i=1}^{n} c_i \otimes f_i) = \sum_{i=1}^{n} c_i f_i = 0$, where n is the minimal number of nonzero summands. If $n = 1$, then $c_1 f_1 = 0$ implies that $c_1 \otimes f_1 = 0$ because of (2). If $n > 1$, let $g \in P[M]$ be arbitrary. By (3), the elements of C and $P[M]$ commute. Hence $\mu(\sum_{i=2}^{n} c_i \otimes (f_i g f_1 - f_1 g f_i)) = 0$. Since this element of the kernel has $n-1$ summands we must have

$$f_i g f_1 = f_1 g f_i,\ i \in \{1, \dots, n\},\ g \in P[M]. \tag{2.1}$$

Fix an ordering $<$ of the monoid M such that $(M, <)$ is an ordered monoid. Let f_{i0}, g_0 be the greatest elements of $\operatorname{supp} f_i$ and $\operatorname{supp} g$ respectively. Then (2.1) implies that

$$f_{i0} m f_{10} = f_{10} m f_{i0},\ i \in \{1, \dots, n\},\ m \in M. \tag{2.2}$$

Now we look at (2.2) as an equality in $H(M)$. Letting $m = 1$, $f_{10}^{-1} f_{i0} = f_{i0} f_{10}^{-1}$ for $i \in \{1, \dots, n\}$. This and (2.2) tells us that $f_{10}^{-1} f_{i0}$ is in the center of $H(M)$ for $i \in \{1, \dots, n\}$. Hence $f_{10} = f_{i0}$ for $i \in \{1, \dots, n\}$ because $H(M)$ has trivial center. Therefore, for some $m_i \in P$, the greatest element in $\operatorname{supp}(f_i - m_i f_1)$ is smaller than f_{i0}, but we still have $(f_i - m_i f_1) g f_1 = f_1 g (f_i - m_i f_1)$ for each g and $i = 1, \dots, n$. Thus $f_i = m_i f_1$ for each i and $\sum_{i=1}^{n} c_i \otimes f_i = (\sum_{i=1}^{n} c_i m_i) \otimes f_1$ reducing to the case of $n = 1$. This shows that $\ker \mu = 0$. □

Suppose that R is a domain (a ring without zero divisors), that C is a central subfield of R, and that M is either the free group or the free monoid. The free group is known to be orderable, see for example [12]. Hence all conditions in Lemma 1 are satisfied. Thus we obtain Lemma 2. It is [26, Lemma 1] and [16, Lemma 2.1].

Lemma 2. *Let R be a domain with prime subfield P. Let C be any central subfield of R. Let M be a free submonoid (subgroup) of $U(R)$. Then the algebra generated by P and M is the monoid (group) algebra $P[M]$ if and only if the algebra generated by C and M is the monoid (group) algebra $C[M]$.* □

Our next result shows that there is a free group algebra inside the Malcev-Neumann skew field of fractions provided there is a free monoid inside the ordered group.

Let $X = \{x_i\}_{i \in I}$ be a set, M the free monoid on X and H the free group on X. If R is a ring, $R\langle\langle X\rangle\rangle = \{\sum_{w \in M} w r_w \mid r_w \in R \text{ for all } w \in M\}$ is a ring with the natural sum and the product induced from the operations of R and M, together with the relations $rw = wr$, for all $r \in R$ and $w \in M$. It has been proved in [1, Section 2] that the group ring $R[H]$ embeds in the power series ring via the morphism of R-rings $\Upsilon\colon R[H] \to R\langle\langle X\rangle\rangle$ determined by $\Upsilon(x_i) = 1 + x_i$ for all $i \in I$. For the case of $R = \mathbb{Z}$ this result was proved in [11, Theorem 4.3] and as noted in [1], the same proof of Fox works also with any ring R.

Lemma 3. *Let $(G,<)$ be an ordered group. Let X be a subset of G consisting of positive elements and such that X is the basis of a noncommutative free submonoid of G. Let K be a skew field and $K[\bar{G};\sigma,\tau]$ a crossed product. Then $K(\bar{G};\sigma,\tau)$ contains a noncommutative free group algebra over its center Z. More precisely, $\{1+\bar{x}\mid x\in X\}$ is a basis of a free group H, and the free group algebra $Z[H]$ embeds in $K(\bar{G};\sigma,\tau)$.*

Proof. Let P be the prime subfield of K. For any finite subset Y of X, let M_Y denote the free submonoid of G generated by Y. Note that $(1-(\sum_{x\in Y}\bar{x}))^{-1}=\sum_{w\in M_Y}\bar{w}\zeta_w\in K((\bar{G};\sigma,\tau,<))$ where $\zeta_w\in K\setminus\{0\}$ for each $w\in M_Y$. Hence M_Y is a well-ordered subset of G. Moreover, it is easy to prove by induction on the lengths that $\bar{w}_1\zeta_{w_1}\bar{w}_2\zeta_{w_2}=\overline{w_1w_2}\zeta_{w_1w_2}$ for all $w_1,w_2\in M_Y$. Thus Y is the basis of a free monoid. The elements of P commute with the elements of $\{\bar{w}\zeta_w\mid w\in N\}$, we obtain that the power series rings $P\langle\langle Y\rangle\rangle$ embeds in $K((\bar{G};\sigma,\tau,<))$ via the natural morphism of P-algebras defined by $x\mapsto\bar{x}$, $x\in Y$. If we set H_Y to be the free group on Y, then $P[H_Y]$ embeds in $K((\bar{G};\sigma,\tau,<))$ via $x\mapsto 1+\bar{x}$, $x\in Y$. Observe that $1+\bar{x}\in K(\bar{G};\sigma,\tau)$, and therefore $P[H_Y]$ embeds in $K(\bar{G};\sigma,\tau)$. Since this can be done for any finite set Y of X, we obtain that in fact $P[H]$ embeds in $K(\bar{G};\sigma,\tau)$. By Lemma 2, the free group algebra $Z[H]$ embeds in $K(\bar{G};\sigma,\tau)$. $\square$

We note that K could be any ring and Z the centralizer of $\{\bar{x}\mid x\in X\}$ in Lemma 3

We say that a group G is *Ore embeddable* if the ring $K[\bar{G};\sigma,\tau]$ is an Ore domain for each skew field K and crossed product $K[\bar{G};\sigma,\tau]$. Examples of Ore embeddable groups are torsion-free abelian groups and torsion-free nilpotent groups.

Lemma 4. *Let R be a ring with a skew field of fractions $\iota\colon R\hookrightarrow F$. Let G be a group. Consider a crossed product $R[\bar{G};\sigma,\tau]$. Suppose that the automorphism $\sigma(x)\in\operatorname{Aut}(R)$ extends to a (unique) automorphism $\varsigma(x)\in\operatorname{Aut}(F)$ for each $x\in G$. Then*

$$\kappa\colon R[\bar{G};\sigma,\tau]\hookrightarrow F[\bar{G};\varsigma,\tau]$$

by extension of scalars, that is, $\kappa_{|R}=\iota$ and $\kappa(\bar{x})=\bar{x}$.

If, moreover, R is an Ore domain, and G is an Ore embeddable group, then $R[\bar{G};\sigma,\tau]$ is an Ore domain with the same Ore skew field of fractions as $F[\bar{G};\varsigma,\tau]$.

Proof. First of all we construct $F[\bar{G};\varsigma,\tau]$. Consider a free F-module T with basis $\bar{G}$, a copy of G. Then each element of T is a finite sum of the form $\sum_{x\in G}\bar{x}a_x$, where $a_x\in F$ for each $x\in G$. Define the product map $\prod\colon T\times T\to T$ by

$$\Big(\sum_{x\in G}\bar{x}a_x,\sum_{x\in G}\bar{x}b_x\Big)\longmapsto\sum_{x\in G}\bar{x}\Big(\sum_{yz=x}\tau(y,z)a_y^{\varsigma(z)}b_z\Big).$$

Notice that κ defines a right R-module embedding of $R[\bar{G};\sigma,\tau]$ inside T. Moreover, $\prod$ restricted to (the image of) $R[\bar{G};\sigma,\tau]$ is the product in $R[\bar{G};\sigma,\tau]$. Now, since ι is an epimorphism of rings [7, Proposition 4.1.1], the conditions of [30, Lemma 1.1] are satisfied for ς and τ. This shows that T is a ring with product defined by the map $\prod$, and $T=F[\bar{G};\varsigma,\tau]$. Then clearly $R[\bar{G};\sigma,\tau]$ embeds in $F[\bar{G};\varsigma,\tau]$ via κ, as desired.

Suppose that R is an Ore domain. By the universal property of Ore domains, any automorphism of R can be extended to its Ore field of fractions F. Since G is Ore embeddable, $F[\bar{G};\varsigma,\tau]$ has an Ore skew field of fractions. Hence $R[\bar{G};\sigma,\tau]$ embeds in $Q_{cl}(F[\bar{G};\varsigma,\tau])$, the Ore skew field of fractions of $F[\bar{G};\varsigma,\tau]$. A classical argument shows that the elements of $Q_{cl}(F[\bar{G};\varsigma,\tau])$ are of the form $g^{-1}f$ (fg^{-1}) for certain $f,g\in R[\bar{G};\sigma,\tau]$ (see for example [19, Theorem 10.28]). $\square$

Following [8], let I be a set. Suppose that there is a map $I \to R((\bar{G};\sigma,\tau,<))$, given by $i \mapsto f_i = \sum_{x\in G} \bar{x}a_{ix}$, such that the following two conditions hold:

(1) $\bigcup\limits_{i\in I} \operatorname{supp}(f_i)$ is well ordered,
(2) for each $x \in G$ the set $\{i \in I \mid x \in \operatorname{supp}(f_i)\}$ is finite.

Then we say that $\sum_{i\in I} f_i$ is *defined in* $R((\bar{G};\sigma,\tau,<))$. In this situation $\sum_{i\in I} f_i$ will be used to denote $\sum_{x\in G} \bar{x}\Big(\sum_{\{i\mid x\in \operatorname{supp} f_i\}} a_{ix}\Big)$. Note that $\sum_{i\in I} f_i$ is then an element of $R((\bar{G};\sigma,\tau,<))$.

Let I, J be sets, and let $\sum\limits_{i\in I} f_i$, $\sum\limits_{i\in I} g_i$, $\sum\limits_{j\in J} h_j$ be defined in $R((\bar{G};\sigma,\tau,<))$. As can be seen in [8], the following hold true:

(i) For any $a \in R$, $\sum\limits_{i\in I} f_i a$ is defined in $R((\bar{G};\sigma,\tau,<))$ and equals $\Big(\sum\limits_{i\in I} f_i\Big)a$.
(ii) $\sum\limits_{i\in I}(f_i+g_i)$ is defined in $R((\bar{G};\sigma,\tau,<))$ and equals $\sum\limits_{i\in I} f_i + \sum\limits_{i\in I} g_i$.
(iii) $\sum\limits_{(i,j)\in I\times J}(f_ih_j)$ is defined in $R((\bar{G};\sigma,\tau,<))$ and equals $(\sum\limits_{i\in I} f_i)(\sum\limits_{j\in J} h_j)$.

Now we use this to prove that Malcev-Neumann series ring behave very much like crossed products.

Proposition 5. *Let K be a skew field and $(G,<)$ an ordered group. Consider a crossed product $K[\bar{G};\sigma,\tau]$ and its Malcev-Neumann series ring $K((\bar{G};\sigma,\tau,<))$. Let N be a normal subgroup of G. The following statements hold true.*

(1) *The crossed product structure of $K[\bar{G};\sigma,\tau] = K[\bar{N};\sigma,\tau][\overline{G/N};\tilde{\sigma},\tilde{\tau}]$ extends to $K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$, and to $K((\bar{N};\sigma,\tau,<))[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$ in the natural way. Therefore*

$$\begin{aligned} K[\bar{N};\sigma,\tau][\overline{G/N};\tilde{\sigma},\tilde{\tau}] &\hookrightarrow K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}] \\ &\hookrightarrow K((\bar{N};\sigma,\tau,<))[\overline{G/N};\tilde{\sigma},\tilde{\tau}] \\ &\hookrightarrow K((\bar{N};\sigma,\tau,<))((\overline{G/N};\tilde{\sigma},\tilde{\tau},\prec)). \end{aligned}$$

(2) *If the group G/N is Ore embeddable, then $K(\bar{G};\sigma,\tau)$ is the Ore field of fractions of $K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$.*
(3) *If N is a convex normal subgroup of G, then moreover*
  (i) *$K((\bar{G};\sigma,\tau,<)) = K((\bar{N};\sigma,\tau,<))((G/N;\tilde{\sigma},\tilde{\tau},\prec))$.*
  (ii) *$K(\bar{G};\sigma,\tau) = K(\bar{N};\sigma,\tau)(\overline{G/N};\tilde{\sigma},\tilde{\tau})$, that is, $K(\bar{G};\sigma,\tau)$ is the Malcev-Neumann skew field of fractions of $K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$.*

Proof. Fix a transversal $\{x_\alpha \mid \alpha \in G/N\}$ of N in G with $x_1 = 1$. Set $\bar{\alpha} = x_\alpha$ for each $\alpha \in G/N$. Set $S = K[\bar{G};\sigma,\tau]$. Consider the structure of $K[\bar{G};\sigma,\tau]$ as a crossed product $S[G/N;\tilde{\sigma},\tilde{\tau}]$ as in Section 1.2.

Given $\alpha \in G/N$, $\tilde{\sigma}(\alpha)\colon S \to S$ is defined by $y \mapsto \bar{x}_\alpha^{-1}y\bar{x}_\alpha$. It can be clearly extended to an automorphism of $K((\bar{N};\sigma,\tau,<))$. We now prove that $\tilde{\sigma}(\alpha)$ can also be extended to an automorphism of $K(\bar{N};\sigma,\tau)$. First note that $K(\bar{N};\sigma,\tau) = \cup_{i\geq 0}S_i$ where $S_0 = S$, and for $i \geq 0$, S_{i+1} is the subring generated by S_i and the inverses of the nonzero elements of S_i. Of course $\tilde{\sigma}(\alpha)$ gives an isomorphism of S_0. Suppose that i is such that $\tilde{\sigma}(\alpha)$ gives an isomorphism of S_i. Now notice that if $f \in S_i \setminus \{0\}$, $\bar{x}_\alpha^{-1}f^{-1}\bar{x}_\alpha = (\bar{x}_\alpha^{-1}f\bar{x}_\alpha)^{-1} \in S_{i+1}$. Hence $\tilde{\sigma}(\alpha)(S_{i+1}) \subseteq S_{i+1}$, and clearly it is injective. Also $f^{-1} = \bar{x}_\alpha^{-1}(\bar{x}_\alpha f^{-1}\bar{x}_\alpha^{-1})\bar{x}_\alpha = \bar{x}_\alpha^{-1}(\tilde{\sigma}(\alpha)^{-1}(f))^{-1}\bar{x}_\alpha \in S_{i+1}$. Therefore, $\tilde{\sigma}(\alpha)$ can be seen as an isomorphism of $K(\bar{N};\sigma,\tau)$. We will denote these extensions of $\tilde{\sigma}(\alpha)$ again as $\tilde{\sigma}(\alpha)$

Note that the elements of the set $\{\bar{\alpha} \mid \alpha \in G/N\}$ are linearly independent over $K((\bar{N};\sigma,\tau,<))$, and hence over $K(\bar{N};\sigma,\tau)$.

Let $A=\sum_{\alpha\in G/N} f_\alpha\bar{\alpha}$ and $B=\sum_{\alpha\in G/N} g_\alpha\bar{\alpha}$ be defined in $K((\bar{G};\sigma,\tau,<))$ where $f_\alpha, g_\alpha\in K((\bar{N};\sigma,\tau,<))$ (or $f_\alpha,g_\alpha\in K(\bar{N};\sigma,\tau)$) for all $\alpha\in G/N$. Then, by the discussion previous to the statement of Proposition 5,

$$A+B=\sum_{\alpha\in G/N}\bar{\alpha}(f_\alpha+g_\alpha),\quad A\cdot B=\sum_{\alpha\in G/N}\bar{\alpha}\Big(\sum_{\beta\gamma=\alpha}\tilde{\tau}(\beta,\gamma)f_\beta^{\tilde{\sigma}(\gamma)}g_\gamma\Big). \tag{2.3}$$

Observe that if the sets $\operatorname{supp}_{G/N}A=\{\alpha\mid f_\alpha\neq 0\}$ and $\operatorname{supp}_{G/N}B=\{\alpha\mid g_\alpha\neq 0\}$ are finite, then A and B are always defined in $K((\bar{G};\sigma,\tau,<))$. Therefore (1) is proved.

To prove (2), note that $K(\bar{G};\sigma,\tau)$ is the skew subfield of $K((\bar{G};\sigma,\tau,<))$ generated by $S[G/N;\tilde{\sigma},\tilde{\tau}]$. The skew field $K(\bar{G};\sigma,\tau)$ clearly contains $K(\bar{N};\sigma,\tau)$ and the set $\{\bar{\alpha}\mid\alpha\in G/N\}$, thus it contains $K(\bar{N};\sigma,\tau)[G/N;\tilde{\sigma},\tilde{\tau}]$, and hence $K(\bar{G};\sigma,\tau)$ contains the Ore field of fractions of $K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$. On the other hand, $K(\bar{N};\sigma,\tau)[\overline{G/N};\tilde{\sigma},\tilde{\tau}]$ contains $S[\overline{G/N};\tilde{\sigma},\tilde{\tau}]=K[\bar{G};\sigma,\tau]$.

Now we prove (3)(i). For each $x\in G$ there exist unique $\alpha\in G/N$ and $n_\alpha\in N$ such that $x=x_\alpha n$.

Let $f=\sum_{x\in G}\bar{x}a_x\in K((\bar{G};\sigma,\tau,<))$ where $a_x\in K$ for each $x\in G$. For each $x\in G$, $\bar{x}a_x=\bar{\alpha}\bar{n}\zeta_{n\alpha}$ for some $\zeta_{n\alpha}\in K$. For each $\alpha\in G/N$, let $f_\alpha=\sum_{n\in N}\bar{n}\zeta_{n\alpha}$. Then $f_\alpha\in K((\bar{N};\sigma,\tau,<))$. Otherwise if $n_1>n_2>\cdots>n_r>\cdots$ is a strictly descending chain of elements in $\operatorname{supp} f_\alpha$, then $x_\alpha n_i\in\operatorname{supp} f$, and $x_\alpha n_1>x_\alpha n_2>\cdots>x_\alpha n_r>\cdots$, a contradiction. Also, the set $\{\alpha\in G/N\mid f_\alpha\neq 0\}$ is well ordered. Otherwise, if $\alpha_1>\alpha_2>\cdots>\alpha_r>\cdots$, there is $x_i=x_{\alpha_i}n_i\in\operatorname{supp} f$. Then, by the convexity of N, $g_1>g_2>\cdots>g_r>\cdots$, a contradiction.

Hence $\sum_{\alpha\in G/N}\bar{\alpha}f_\alpha$ is defined in $K((\bar{G};\sigma,\tau,<))$ and equals f.

On the other hand, given a well ordered set $\Delta\subseteq G/N$, and series $f_\alpha=\sum_{n\in N}\bar{n}\zeta_{n\alpha}\in K((\bar{N};\sigma,\tau,<))$ for each $\alpha\in\Delta$. Then it is not very difficult to see that the series $\sum_{\alpha\in\Delta}\bar{\alpha}f_\alpha$ is defined in $K((\bar{G};\sigma,\tau,<))$. Therefore $K((\bar{G};\sigma,\tau,<))=K((\bar{N};\sigma,\tau,<))((\overline{G/N};\tilde{\sigma},\tilde{\tau},\prec))$ as sets. But (2.3) shows that they are equal as rings.

To show (3)(ii) proceed as in the proof of (2). $\square$

We remark that Proposition 5(3)(ii) is an important particular case of [18], but the proof here is much easier.

3. Main result

The next theorem is the main result of the paper. This section is devoted to prove it.

Theorem 6. *Let $(G,<)$ be a noncommutative ordered group. Let K be a skew field, and let $K[G]$ be the group ring over K. Then $K(G)$ contains noncommutative free group algebras over its center.*

The proof of Theorem 6 is divided in three parts, depending on which kind of ordered group $(G,<)$ is. We will single out three types of ordered groups.

- Type 1: Every convex jump in $(G,<)$ is central, that is, $[H,G]\subseteq N$ for all convex jumps (N,H) of $(G,<)$.
- Type 2: Every convex subgroup of $(G,<)$ is normal in G, but there exists a convex jump (N,H) which is not central, that is, $[H,G]\nsubseteq N$.
- Type 3: There exists a convex subgroup of $(G,<)$ which is not normal in G.

We note that any ordered group $(G,<)$ falls into one (and only one) of these three types of ordered groups. This is because if $(G,<)$ is of type 1, then every convex subgroup is normal in G as noted in [3, p. 226]. Indeed, let U be a convex subgroup

of $(G,<)$, $u\in U$ and $x\in G$. Consider the convex jump (N_u,H_u) determined by u. Then $N_u\subset H_u\subseteq U$, and $x^{-1}ux=u[u,x]\in U$ since $[H_u,G]\subseteq N_u$ by hypothesis.

On the other hand it may happen that there exist two total orders $<_1$, $<_2$ of an orderable group G such that $(G,<_1)$ and $(G,<_2)$ are ordered groups of different types.

3.1. **Ordered groups of Type 1.** The following are sufficient conditions to extend morphisms of rings $\varphi\colon R_1\to R_2$ and morphisms of groups $\eta\colon G_1\to G_2$ to morphisms of rings $\Phi\colon R_1[\bar G_1;\sigma_1,\tau_1]\to R_2[\bar G_2;\sigma_2,\tau_2]$ and $\Phi\colon R_1((\bar G_1;\sigma_1,\tau_1,<))\to R_2((\bar G_2;\sigma_2,\tau_2,<))$.

Lemma 7. *Let $\varphi\colon R_1\to R_2$ be a morphism of rings, and $\eta\colon G_1\to G_2$ a morphism of groups. Consider crossed product group rings $R_1[\bar G_1;\sigma_1,\tau_1]$ and $R_2[\bar G_2;\sigma_2,\tau_2]$. Suppose that*

$$\varphi(r^{\sigma_1(x)}) = \varphi(r)^{\sigma_2(\eta(x))}, \tag{3.1}$$
$$\varphi(\tau_1(x,y)) = \tau_2(\eta(x),\eta(y)). \tag{3.2}$$

Then the following hold true

(1) *The map $\Phi\colon R_1[\bar G_1;\sigma_1,\tau_1]\to R_2[\bar G_2;\sigma_2,\tau_2]$, defined by*
$$\Phi\Big(\sum_{x\in G_1}\bar x a_x\Big)=\sum_{x\in G_1}\overline{\eta(x)}\varphi(a_x), \tag{3.3}$$
is a morphism of rings.

(2) *If, moreover, $(G_1,<_1)$, $(G_2,<_2)$ are ordered groups and $\eta\colon G_1\to G_2$ is an injective morphism of ordered groups. Then $\Phi\colon R_1((\bar G_1;\sigma_1,\tau_1,<_1))\to R_2((\bar G_2;\sigma_2,\tau_2,<_2))$, the natural extension of Φ in* (1)*, defined by*
$$\Phi\Big(\sum_{x\in G_1}\bar x a_x\Big)=\sum_{x\in G_1}\overline{\eta(x)}\varphi(a_x), \tag{3.4}$$
is a morphism of rings.

Proof. The map Φ in (2) is well defined because η is an injective morphism of ordered groups. Now the same proof works for (1) and (2). By definition, $\Phi(1)=\Phi(\bar 1)=\overline{\eta(1)}=\bar 1=1$. Let $\sum_{x\in G_1}\bar x a_x$, $\sum_{x\in G_1}\bar x b_x\in R_1((\bar G_1;\sigma_1,\tau_1,<_1))$. Then

$$\begin{aligned}
\Phi\Big(\Big(\sum_{x\in G_1}\bar x a_x\Big)\Big(\sum_{x\in G_1}\bar x b_x\Big)\Big) &= \Phi\Big(\sum_{x\in G_1}\bar x\Big(\sum_{yz=x}\tau_1(y,z)a_y^{\sigma_1(z)}b_z\Big)\Big)\\
&= \sum_{x\in G_1}\overline{\eta(x)}\Big(\sum_{yz=x}\varphi\big(\tau_1(y,z)\big)\varphi(a_y^{\sigma_1(z)})\varphi(b_z)\Big)\\
&= \sum_{x\in G_1}\overline{\eta(x)}\Big(\sum_{yz=x}\tau_2\big(\eta(y),\eta(z)\big)\varphi(a_y)^{\sigma_2(\eta(z))}\varphi(b_z)\Big)\\
&= \Big(\sum_{x\in G_1}\overline{\eta(x)}\varphi(a_x)\Big)\Big(\sum_{x\in G_1}\overline{\eta(x)}\varphi(b_x)\Big)\\
&= \Phi\Big(\sum_{x\in G_1}\bar x a_x\Big)\Phi\Big(\sum_{x\in G_1}\bar x b_x\Big).
\end{aligned}$$

Analogously, it can be seen that Φ is additive. □

The trivial situation where the crossed products $R_1[\bar G_1;\sigma_1,\tau_1]$ and $R_2[\bar G_2;\sigma_2,\tau_2]$ are group rings is well known: for any morphism of rings $\varphi\colon R_1\to R_2$ and any morphism of groups $\eta\colon G_1\to G_2$ conditions (3.1) and (3.2) are satisfied because σ_1, σ_2 and τ_1, τ_2 are trivial.

Although, as pointed out in [22, p. 674], the proof of [9, Proposition 3.1] is not correct, some minor changes give Example 8. It sets the situation where we will use Lemma 7 and it also fixes the notation that will be used in what follows.

Example 8. Let G be a group with a normal subgroup N. Let K be a skew field. As in Section 1.2, we look at the group ring $K[G]$ as a crossed product $S[G/N; \tilde{\sigma}, \tilde{\tau}]$ where $S = K[N]$. Fix a transversal $\{x_\alpha \mid \alpha \in G/N\}$ of N in G with $x_1 = 1$. In our situation the maps $\tilde{\sigma}$ and $\tilde{\tau}$ are defined as follows. For each $\alpha \in G/N$, $\tilde{\sigma}(\alpha)$ is given by $x_\alpha^{-1} r x_\alpha$ for all $r \in S$; for each $\alpha, \beta \in G/N$, $\tilde{\tau}(\alpha, \beta) = n_{\alpha\beta}$ where $n_{\alpha\beta}$ is the unique element in N such that $x_{\alpha\beta} n_{\alpha\beta} = x_\alpha x_\beta$.

Let $R_1 = S$, $R_2 = K$, $G_1 = G_2 = G/N$, $\eta\colon G/N \to G/N$ be the identity, and $\varphi\colon S \to K$ the augmentation map. Then there exists a morphism $\Phi\colon K[G] = S[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}] \to K[G/N]$ extending φ and η. Indeed,

$$\varphi(\tilde{\tau}(\alpha, \beta)) = \varphi(n_{\alpha\beta}) = 1 = \tau_2(\eta\alpha, \eta\beta),$$

$$\varphi(r^{\tilde{\sigma}(\alpha)}) = \varphi(x_\alpha r x_\alpha^{-1}) = \varphi(r) = \varphi(r)^{\sigma_2(\alpha)},$$

for all $\alpha, \beta \in G/N$ and $r \in S$. Thus conditions (3.1) and (3.2) are satisfied.

Suppose moreover that N is such that G/N is orderable. If $(G/N, \prec)$ is an ordered group, then Φ can be extended to $\Phi\colon S((G/N; \tilde{\sigma}, \tilde{\tau}, \prec)) \to K((G/N; \prec))$ by Lemma 7(2).

Given a series $A = \sum_{\alpha \in G/N} \alpha a_\alpha \in K((G/N; \prec))$, we define the *good preimage* $\hat{A}$ of A by Φ as

$$\hat{A} = \sum_{\alpha \in G/N} \bar{\alpha} a_\alpha \in S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec)).$$

Note that if A is invertible in $K((G/N; \prec))$, then $\hat{A}$ is invertible in $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$ because $a_{\alpha_0} \in K$ is invertible in S, where $\alpha_0 = \min \operatorname{supp} A = \min \operatorname{supp} \hat{A}$.

Let P be a central subfield of K. Then it is a central subfield of $K((G/N; \prec))$ and $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$. Let I be a set, and $A_i, B_i \in K((G/N; \prec)) \setminus \{0\}$ for each $i \in I$. Suppose that $\{A_i B_i^{-1}\}_{i \in I}$ generate a free group algebra over P inside $K((G/N; \prec))$. Let $\{\hat{A}_i, \hat{B}_i\}_{i \in I}$ be the set of good preimages of $\{A_i, B_i\}_{i \in I}$. For each $i \in I$, $\hat{A}_i \hat{B}_i^{-1}$ is an invertible element of $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$. Then $\{\hat{A}_i \hat{B}_i^{-1}\}_{i \in I}$ generate a free group algebra over P inside $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$ because of the following reasons: Φ is a morphism of rings such that $\Phi(\hat{A}_i \hat{B}_i^{-1}) = A_i B_i^{-1}$ for all $i \in I$, the series $\hat{A}_i \hat{B}_i^{-1}$ is invertible in $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$ for all $i \in I$, $\Phi(a) = a$ for all $a \in P$, and the elements of P commute with the elements of $\{\hat{A}_i \hat{B}_i^{-1}\}_{i \in I}$. □

Proposition 9. *Let $(G, <)$ be an ordered group with a normal subgroup N such that G/N is a noncommutative torsion-free nilpotent group. Let K be a skew field with prime subfield P. Consider the group ring $K[G]$ and the Malcev-Neumann series ring $K((G; <))$. Then $K(G)$ contains a noncommutative free group P-algebra.*

Proof. Recall that if H is a nilpotent group and L a skew field then any crossed product $L[\bar{H}; \sigma, \tau]$ is an Ore domain. We denote its Ore skew field of fractions by $Q_{cl}(L[\bar{H}; \sigma, \tau])$. Thus $Q_{cl}(L[\bar{H}; \sigma, \tau])$ embeds in any skew field that contains $L[\bar{H}; \sigma, \tau]$.

Set $S = K[N]$. As in Example 8, $K[G] = S[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}]$. By Proposition 5(1), $S[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}] \hookrightarrow K(N)[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}]$. Moreover, $K(G)$ is the Ore field of fractions of $K(N)[\overline{G/N}; \tilde{\sigma}, \tilde{\tau}]$ by Proposition 5(2).

Fix an order $\prec$ on G/N such that $(G/N, \prec)$ is an ordered group. The foregoing embedding can be extended to $S((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec)) \hookrightarrow K(N)((\overline{G/N}; \tilde{\sigma}, \tilde{\tau}, \prec))$. Hence

we obtain the following commutative diagram of ring embeddings

$$\begin{tikzcd}
S[\overline{G/N};\tilde\sigma,\tilde\tau] \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & K(N)[\overline{G/N};\tilde\sigma,\tilde\tau] \arrow[dl, hook'] \arrow[dd, hook] \\
& K(G) \arrow[dr, hook] & \\
S((\overline{G/N};\tilde\sigma,\tilde\tau,\prec)) \arrow[rr, hook] & & K(N)((\overline{G/N};\tilde\sigma,\tilde\tau,\prec))
\end{tikzcd} \tag{3.5}$$

By [10, Corollary 2.1], $Q_{cl}(P[G/N])$ contains a noncommutative free group P-algebra. Since $P[G/N]$ is a subring of $K[G/N]$, $Q_{cl}(P[G/N]) \subseteq Q_{cl}(K[G/N])$. Hence there exist $A_1, B_1, A_2, B_2 \in K[G/N]$ such that $A_1B_1^{-1}, A_2B_2^{-1}$ generate a noncommutative free group P-algebra inside $Q_{cl}(K[G/N]) \subseteq K((G/N;\prec))$. Consider the morphism of rings $\Phi\colon S((\overline{G/N};\tilde\sigma,\tilde\tau,\prec)) \to K((G/N;\prec))$ constructed in Example 8. Let $\hat A_i, \hat B_i \in S((\overline{G/N};\tilde\sigma,\tilde\tau,\prec))$, $i = 1, 2$, be good preimages of A_i, B_i as in Example 8. Hence, by Example 8, they generate a noncommutative free group P-algebra inside $S((\overline{G/N};\tilde\sigma,\tilde\tau,\prec))$. Observe that $\hat A_i, \hat B_i \in K[G]$, and therefore, this free group algebra is also inside $K(G)$ by diagram (3.5). $\square$

Proof of Theorem 6 for ordered groups of Type 1. Suppose that $(G, <)$ is a noncommutative ordered group of Type 1.

Let x, y be positive elements of $(G, <)$ such that $[x, y] \neq 1$. If $z = \min\{x, y\}$ and (N_z, H_z) is the convex jump determined by z, then $[x, y] \in N_z$ since every convex jump is central. Moreover, neither x nor y belong to N_z.

Let (N, H) be the convex jump determined by $[x, y]$. Hence $[x, y] \in H \setminus N$. Moreover, since $H \subseteq N_z$, neither x nor y belong to H.

Let A be the subgroup of G generated by $\{x, y\}$, and let $B = N \cap A$. First observe that $A/B \hookrightarrow G/N$ which proves that A/B is torsion free because G/N is an orderable group. Since $[x, y] \notin N$, A/B is not commutative. On the other hand $A/(H \cap A)$ is a nontrivial commutative group because $[x, y] \in H$. Hence $[A, A] \subseteq H \cap A$. Therefore, since all convex jumps are central, $[[A, A], A] \subseteq N \cap A = B$. Therefore A/B is a torsion-free nilpotent group of index two. By Proposition 9, there exist noncommutative free group algebras over the prime subfield P of K inside $K(A) \subseteq K(G)$. Now Lemma 2 implies the result. $\square$

3.2. **Ordered groups of Type 2.** The strategy to prove Theorem 6 in this case is to find a noncommutative free monoid inside ordered groups of type 2. The following key result provides a useful criterion for finding free monoids inside groups. It is [32, Corollary 2.5].

Lemma 10. *A group G contains a free monoid on two generators if and only if there exist a nonempty subset $A \subseteq G$ and elements $a, b \in G$ such that $aA \cup bA \subseteq A$ and $aA \cap bA = \emptyset$. In this case a, b generate a free monoid.* $\square$

The following is a slight modification of [32, Lemma 4.16].

Lemma 11. *Let $r \in \mathbb{R}$ with $r \geq 2$ or $0 < r \leq \frac{1}{2}$. Let there be integers $0 \leq a_1 < a_2 < \cdots < a_n$ and $0 \leq b_1 < b_2 < \cdots < b_m$. If*

$$r^{a_1} + r^{a_2} + \cdots + r^{a_n} = r^{b_1} + r^{b_2} + \cdots r^{b_m}, \tag{3.6}$$

then $n = m$ and $(a_1, \ldots, a_n) = (b_1, \ldots, b_m)$.

Proof. First suppose that $r \geq 2$. We proceed by induction on the maximum number of summands in (3.6). If $1 = n \geq m$, the result is clear. If $a_n \neq b_m$, suppose that $a_n > b_m$ (the other case is proved in the same way). Then $r^{a_n} = \sum_{i=1}^{m} r^{b_i} -$

$\sum_{i=1}^{n-1} r^{a_i} \leq \sum_{i=1}^{m} r^{b_i} \leq \sum_{l=0}^{b_m} r^l = \frac{r^{b_m}-1}{r-1} \leq r^{b_m} - 1 \leq r^{a_n} - 1$, a contradiction. Thus $a_n = b_m$. Now we can apply induction to obtain the result.

Suppose now that $0 < \frac{1}{2} \leq r$. If $a_n \geq b_m$, we multiply the expression by r^{-a_n} to obtain $(r^{-1})^0 + (r^{-1})^{(a_n - a_{n-1})} + \cdots + (r^{-1})^{(a_n - a_1)} = (r^{-1})^{(a_n - b_m)} + \cdots + (r^{-1})^{(a_n - b_1)}$ and apply the case $2 \leq r^{-1}$. □

The next lemma is basically proved in [32, Examples 4.19].

Lemma 12. *Let $(H, +)$ be a nontrivial additive sugroup of $(\mathbb{R}, +)$ and $C = \langle x \rangle$ an infinite cyclic group. Consider the semidirect product $G = H \rtimes C$ where x acts on H by multiplication by $r \in \mathbb{R}$, where either $0 < r \leq \frac{1}{2}$ or $r \geq 2$. That is, $xzx^{-1} = rz$ for all $z \in H$. Let $t \in H \setminus \{0\}$, then $\{tx, x\}$ generate a free noncommutative submonoid of G.*

Proof. Let A be the subset of G defined by

$$A = \{(r^{a_1} + \cdots + r^{a_n})t \cdot x^j \mid 0 \leq a_1 < a_2 < \cdots < a_n,\ a_i \in \mathbb{Z},\ n, j \in \mathbb{N}\}.$$

Observe that $tx \cdot (r^{a_1} + \cdots + r^{a_n})t \cdot x^j = (t + x(r^{a_1} + \cdots + r^{a_n})x^{-1}) \cdot x^{j+1} = (1 + r^{a_1+1} + \cdots + r^{a_n+1})t \cdot x^{j+1}$. On the other hand $x \cdot (r^{b_1} + \cdots + r^{b_m})t \cdot x^l = x(r^{b_1} + \cdots + r^{b_m})tx^{-1} \cdot x^{l+1} = (r^{b_1+1} + \cdots + r^{b_m+1})t \cdot x^{l+1}$. Thus $txA \subseteq A$ and $xA \subseteq A$. But by Lemma 11, $txA \cap xA = \emptyset$. Therefore, $\{tx, x\}$ generate a noncommutative free monoid by Lemma 10. □

Proof of Theorem 6 for ordered groups of Type 2. Let $(G, <)$ be an ordered group of Type 2. Let (B, L) be a convex jump of $(G, <)$ such that $[L, G] \nsubseteq B$.

We want to find a noncommutative free monoid generated by two positive elements of $(G, <)$. For this it is enough to find two positive elements of $(G/B, \prec)$ that generate a noncommutative free submonoid of G/B.

Set $H = L/B$ which is an additive subgroup of $(\mathbb{R}, +)$. Since $[L, G] \nsubseteq B$ and $L \lhd G$, there exists $x \in G/B$ such that conjugation by x defines a nonidentity morphism of the ordered group H. Hence x acts on H as multiplication by a positive real number $r \neq 1$. Changing to x^n if necessary, we can suppose that either $r \geq 2$ or $0 < r \leq \frac{1}{2}$. Clearly, if C is the infinite cyclic group generated by x, we obtain that the subgroup generated by H and C is a semidirect product $H \rtimes C$. Then by Lemma 12 we obtain the free monoid of positive elemens inside G/B as desired.

Now apply Lemma 3 to obtain a noncommutative free group algebra inside $K(G)$. □

3.3. **Ordered groups of Type 3.** The proof of Theorem 6 for ordered groups of type 3 follows directly from the following result and Lemma 3. Lemma 13 was first proved in [23, Lemma 8]. Our proof is based on the ordering on the group and exhibits the generators of the free monoid when the order is well known.

Lemma 13. *Let $(G, <)$ be an ordered group with a convex subgroup C which is not normal in G, then G contains a noncommutative free monoid.*

Proof. First recall that for every $x \in G$, xCx^{-1} is a convex subgroup of $(G, <)$. Thus, since the convex subgroups of $(G, <)$ form a chain, there exists $b \in G$ such that $bCb^{-1} \subsetneq C$. Indeed, if $C \subsetneq xCx^{-1}$, then $x^{-1}Cx \subsetneq C$. Notice that $b \notin C$, and if (N_b, H_b) is the convex jump determined by b, $C \subseteq N_b \subsetneq H_b$. Choose $a \in C$, such that $a \notin bCb^{-1}$ and $a > 1$ if $b > 1$ or $a < 1$ in case $b < 1$. Observe that for each $n \geq 0$, $b^{n+1}Cb^{-(n+1)} \subsetneq b^nCb^{-n}$ and $b^nab^{-n} \in b^nCb^{-n} \setminus b^{n+1}Cb^{-(n+1)}$. Let (N_0, H_0) be the convex jump determined by a. Then $H_0 \subseteq C \subsetneq H_b$. For each $n \geq 1$, let (N_n, H_n) be the convex jump determined by b^nab^{-n}. Then $b^nab^{-n} \in H_n$ but

$b^nab^{-n} \notin N_n$. Hence, by definition of H_{n+1} and N_n, $H_{n+1} \subseteq b^{n+1}Cb^{-(n+1)} \subseteq N_n$. Therefore $H_{n+1} \subsetneq H_n$ for all $n \geq 0$.

We claim that $\{a, b\}$ generate a noncommutative free monoid. Suppose that $\{a, b\}$ satisfy a positive word. We can suppose that

$$a^{n_1}b^{m_1} \cdots a^{n_s}b^{m_s} = b^{l_1}a^{k_2}b^{l_2} \cdots a^{k_t}b^{l_t} \tag{3.7}$$

with $m_i, n_i \geq 0$, $l_j, k_j \geq 0$ for all i, j and $n_1 \geq 1$, $l_1 \geq 1$. Then (3.7) can be written as

$$\begin{gathered} a^{n_1}(b^{m_1}a^{n_2}b^{-m_1}) \cdots (b^{m_1+\cdots+m_{s-1}}a^{n_s}b^{-(m_1+\cdots+m_{s-1})})b^{m_1+\cdots+m_s} \\ = \\ (b^{l_1}a^{k_2}b^{-l_1}) \cdots (b^{l_1+\cdots+l_{t-1}}a^{k_t}b^{-(l_1+\cdots+l_{t-1})})b^{l_1+\cdots+l_t} \end{gathered} \tag{3.8}$$

Set $l = l_1 + \cdots + l_t$ and $m = m_1 + \cdots + m_s$. Suppose that $l < m$. Then $b^{m-l} \in H_0$ by (3.8) because $a^{n_1} \in H_0$, $b^{(m_1+\cdots+m_i)}a^{n_{i+1}}b^{-(m_1+\cdots+m_i)} \in H_0$ for all $1 \leq i \leq s-1$ and $b^{l_1+\cdots+l_j}a^{k_{j+1}}b^{-(l_1+\cdots+l_j)} \in H_0$ for all $1 \leq j \leq t-1$. This is a contradiction because $b^{m-l} \in H_b \setminus H_0$. Analogously we obtain a contradiction if $m < l$. Suppose now that $l = m$. Then (3.8) shows that $a^{n_1} \in H_1$ because $b^{(m_1+\cdots+m_i)}a^{n_{i+1}}b^{-(m_1+\cdots+m_i)} \in H_1$ for all $1 \leq i \leq s-1$ and $b^{l_1+\cdots+l_j}a^{k_{j+1}}b^{-(l_1+\cdots+l_j)} \in H_1$ for all $1 \leq j \leq t-1$. This is a contradiction because $a^{n_1} \in H_0 \setminus H_1$.

Notice now that if $a, b < 1$ generate a free monoid, then $1 < a^{-1}, b^{-1}$ and they also generate a free monoid. Therefore we can suppose that a, b are positive elements of $(G, <)$ that generate a noncommutative free monoid. □

3.4. **Remarks.** There are orderable groups G such that for each ordering $<$ on G, $(G, <)$ is of type 2, and moreover, each noncommutative finitely generated subgroup H of G is also of type 2. For example each ordering $<$ on $B(1, 2) = \{a, b \mid bab^{-1} = a^2\}$ is of type 2, see for example [31, Section 4]. Then it is easy to prove that each noncommutative finitely generated subgroup is of type 2. Hence the argument to find a group algebra when $(G, <)$ is of type 2 is necessary.

Following [3], let $\mathcal{C}$ denote the class of orderable groups in which every order is central i.e. of type 1, and let $\mathcal{C}_2$ be the class of orderable groups in which every order on every two-generator subgroup is central. Then $\mathcal{C}_2$ is not empty because the class of torsion-free locally nilpotent groups is in $\mathcal{C}_2$ [33], and $\mathcal{C}_2 \subsetneq \mathcal{C}$ [3]. Hence the argument to find free group algebras when $(G, <)$ is of type 1 is necessary for the elements in $\mathcal{C}_2$. Although every locally soluble group in $\mathcal{C}_2$ is locally nilpotent [3], there are finitely generated non-locally-soluble groups in $\mathcal{C}_2$ [4]. Also there are non-locally-soluble groups in $\mathcal{C}_2$ which are residually torsionfree nilpotent without noncommutative free subsemigroups [4].

It is known that there are orderable groups G such that every ordered group $(G, <)$ has no normal convex subgroups (see for example [34] and the references there). We do not know whether there exists a group satisfying that every noncommutative finitely generated subgroup (or every two-generator subgroup) inherits this property. Hence it could happen that the argument for ordered groups of type 3 is not necessary because some subgroup of G could be made into an ordered group of type 1 or 2.

4. Some consequences

Corollary 14. *Let G be a noncommutative orderable group. Let K be a skew field, and let $K[G]$ be the group ring over K. If D is any skew field of fractions of $K[G]$ (it may not be the Malcev-Neumann skew field of fractions), then D contains a noncommutative free algebra over its center.*

If, moreover, the center of D is uncountable, then D contains a noncommutative free group algebra over its center.

Proof. If $K[G]$ is not an Ore domain, it is well known that $K[G]$ contains a free algebra over the prime subfield P of K [19, Proposition 10.25], and therefore D contains a noncommutative free algebra over the center of D by Lemma 2.

If $K[G]$ is a right (or left) Ore domain, then D is the Ore field of fractions of $K[G]$. Fix a total order on G such that $(G,<)$ is an ordered group. Then $K[G]$, and thus D, embeds in $K((G;<))$. Hence $D = K(G;<)$ contains a free group algebra by Theorem 6.

By [16, Theorem], if D has uncountable center Z and contains a noncommutative free algebra over Z, then it contains a noncommutative free group algebra over Z. □

Now we show that in fact we have shown something stronger than Theorem 6.

Corollary 15. *Let $(G,<)$ be an orderable group with G not in $\mathcal{C}_2$. Let K be a skew field and $K[\bar{G};\sigma,\tau]$ a crossed product group ring. Then $K(\bar{G};\sigma,\tau)$ contains a free group algebra over its center.*

Proof. If $(G,<)$ is of type 2 or 3, then we have shown in Section 3.2 and in Section 3.3, respectively, that G contains a noncommutative free monoid.

If $(G,<)$ is of type 1, but $G \notin \mathcal{C}_2$, then there exists a subgroup H of G and a total order $<'$ of H such that $(H,<')$ is of type 1 or 2. Hence H contains a noncommutative free monoid.

Now Lemma 3 implies the result. □

We prove the next result with the techniques developed in Section 3.1. When G is a residually torsion-free nilpotent group, Proposition 16 gives [14, Theorem 5] whose original proof relied on [9, Proposition 3.1].

Proposition 16. *Let K be a commutative field of characteristic not 2, and let $(G,<)$ be an ordered group. Let x,y be two noncommuting elements of G and H the subgroup of G that they generate. If there exists a subgroup $N \lhd H$ such that H/N is a noncommutative torsion-free nilpotent group, then, for any $c,d \in K\setminus\{0\}$, the subgroup $\langle\{1+cx, 1+dy\}\rangle$ of $K(G)$ is a noncommutative free group.*

Proof. Since $H \leq G$, $K(H) \subseteq K(G)$. Thus we may suppose that $H = G$ and $N \lhd G$.

We borrow notation from Example 8 and Proposition 9.

Let α_0 and β_0 be the cosets of x and y, respectively, in G/N. As in Example 8, fix a transversal $\{x_\alpha \mid \alpha \in G/N\}$ with $x_1 = 1$ and such that $x_{\alpha_0} = x$ and $y_{\beta_0} = y$.

Suppose that G/N is torsion-free nilpotent of class two. Fix a total order $\prec$ in G/N such that $(G/N,\prec)$ is an ordered group, and consider the group ring $K[G/N]$. Then, by [14, Proposition 20], $\{1+c\alpha_0, 1+d\beta_0\}$ generate a free group inside $Q_{cl}(K[G/N]) \subseteq K((G/N,\prec))$. Observe that $1+c\bar{\alpha}_0 = 1+cx$ and $1+d\bar{\beta}_0 = 1+dy$ are good preimages of $1+c\alpha_0, 1+d\beta_0$ inside $S((\overline{G/N};\tilde{\sigma},\tilde{\tau},\prec))$ and generate a free group. Proceeding as in Proposition 9, one sees that the subgroup generated by $\{1+cx, 1+dy\}$ inside $K(G)$ is a free group.

Now observe that if G/N is torsion-free nilpotent of greater class, there exists $M \lhd G$, such that $N \subseteq M$ and G/M is a torsion free nilpotent of class 2. Choosing an adequate transversal for G/M, the foregoing argument implies the result for $K(G)$. □

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